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New results on group classification of nonlinear diffusion–convection equations

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Abstract
Using a new method and additional (conditional and partial) equivalence transformations, we performed group classification in a class of variable coefficient $(1+1)$-dimensional nonlinear diffusion–convection equations of the general form $f(x)u_t = (D(u)u_x)_x + K(u)u_x$. We obtain new interesting cases of such equations with the density f localized in space, which have non-trivial invariance algebra. Exact solutions of these equations are constructed. We also consider the problem of investigation of the possible local transformations for an arbitrary pair of equations from the class under consideration, i.e. of describing all the possible partial equivalence transformations in this class.

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1. Introduction

The problems of group classification and exhaustive solutions of such problems are not only interesting from the purely mathematical point of view, but also important for applications. In physical models there often exist *a priori* requirements for symmetry groups that follow from physical laws (in particular, from Galilean or relativistic theory). Moreover, modelling differential equations could contain parameters or functions which have been found experimentally and so are not strictly fixed. (It is said that these parameters and functions are arbitrary elements.) At the same time mathematical models have to be simple enough to allow one to analyse and solve them. Solving the problems of group classification makes it possible to accept for the criterion of applicability the following statement: modelling differential equations have to admit a group with certain properties or the most extensive symmetry group from the possible ones.

In this paper we consider a class of variable coefficient nonlinear diffusion–convection equations of the form

$$f(x)u_t = (g(x)D(u)u_x)_x + K(u)u_x \tag{1}$$

where $f = f(x)$, $g = g(x)$, $D = D(u)$ and $K = K(u)$ are arbitrary smooth functions of their variables, $f(x)g(x)D(u) \neq 0$. The linear case of (1) ($D, K = \text{const}$) was studied by Lie [1] in his classification of linear second-order PDEs with two independent variables. (See also a modern treatment of this subject in [2].) This is why we assume below that $(D_u, K_u) \neq (0, 0)$, i.e. (1) is a nonlinear equation.

Using the transformation $\tilde{t} = t$, $\tilde{x} = \int \frac{\mathrm{d}x}{g(x)}$, $\tilde{u} = u$, we can reduce equation (1) to

$$\tilde{f}(\tilde{x})\tilde{u}_{\tilde{t}} = (D(\tilde{u})\tilde{u}_{\tilde{x}})_{\tilde{x}} + K(\tilde{u})\tilde{u}_{\tilde{x}}$$

where $\tilde{f}(\tilde{x}) = g(x)f(x)$ and $\tilde{g}(\tilde{x}) = 1$. (Likewise any equation of form (1) can be reduced to the same form with $\tilde{f}(\tilde{x}) = 1$.) That is why without loss of generality we restrict ourselves to investigation of the equation

$$f(x)u_t = (D(u)u_x)_x + K(u)u_x. \tag{2}$$

In addition to their intrinsic theoretical interest, equations (2) are used to model a wide variety of phenomena in physics, chemistry, mathematical biology etc. For the case $f(x) = 1$ equation (2) describes vertical one-dimensional transport of water in homogeneous non-deformable porous media. When $K(u) = 0$ this equation describes stationary motion of a boundary layer of fluid over a flat plate, a vortex of incompressible fluid in a porous medium for polytropic relations of gas density and pressure. The outstanding representative of the class of equations (2) is the Burgers equation that is the mathematical model for a large number of physical phenomena. (For more details refer to [4–7].)

Investigation of the nonlinear heat equations by means of symmetry methods started in 1959 with Ovsiannikov's work [8] where he studied symmetries of the equation

$$u_t = (f(u)u_x)_x. \tag{3}$$

In 1987 Akhatov, Gazizov and Ibragimov [9] classified the equations

$$u_t = G(u_x)u_{xx}. \tag{4}$$

Dorodnitsyn (in 1982 [10]) performed group classification of the equation

$$u_t = (G(u)u_x)_x + g(u). \tag{5}$$

Oron, Rosenau (in 1986 [11]) and Edwards (in 1994 [12]) presented the most extensive (at the time) list of symmetries of the equations

$$u_t = (G(u)u_x)_x + f(u)u_x. \tag{6}$$

The results of [8, 10, 11] were generalized by Cherniha and Serov (in 1998 [13]) who classified the nonlinear heat equation with a convection term

$$u_t = (G(u)u_x)_x + f(u)u_x + g(u). \tag{7}$$

It should be noted that equations (1)–(7) are particular cases of the more general class of equations

$$u_t = F(t, x, u, u_x)u_{xx} + G(t, x, u, u_x). \tag{8}$$

The group classification of (8) is presented in [17–21]. However, since the equivalence group of (8) is essentially wider than those for (1)–(7) the results of [17–21] cannot be directly used for symmetry classification of equations (1)–(7). Nevertheless, these results are useful for finding additional equivalence transformations in the above classes.

Equations of the form (2) have also been investigated from points of view other than the classic Lie symmetry one. For instance, potential symmetries of subclasses of (2) where e.g. either $f = 1$ or $K = 0$ were studied by Sophocleous [14–16].

Some symmetry properties of class (1) were considered in a recent paper [3]. However, it does not present correct and complete results on the subject. The overwhelming majority of cases with non-trivial Lie symmetry were omitted, and there are mistakes in the cases that were adduced. Nevertheless the subject seemed very interesting, and we decided to study Lie symmetries of class (2).

The ultimate goal of this paper is to present an example of exhaustive solution of the group classification problem in quite a difficult case. After giving a precise definition and a discussion of this problem in the general case, we performed the complete extended group classification and found additional equivalence transformations and exact solutions of equations (2). A lot of new interesting cases of extensions of the maximal Lie symmetry group were obtained for these equations. For example, we determined the equations which have the density f localized in the space of x and are invariant with respect to m-dimensional ($2 \leqslant m \leqslant 4$) Lie symmetry algebras, which allows construction of new exact non-stationary solutions for them.

Problems of general group classification, except for really trivial cases, are very difficult. This can be illustrated by the multitude of papers where such a general classification problem is solved incorrectly or incompletely. There are also many papers on 'preliminary group classification' where authors list some cases with new symmetry but do not claim that the general classification problem is solved completely. For this reason, finding an effective approach to simplification is essentially equivalent to showing the feasibility of solving the problem at all. In this paper we develop and apply a simple and effective tool based on the investigation of the specific compatibility of classifying conditions. This was first proposed in [23] and then applied in solving a number of different group classification problems [24–26]. Another tool is the systematic use of additional (conditional and partial) equivalence transformations, which allows us to put in order, verify and analyse the results obtained.

Our paper is organized as follows. First of all (section 2) we describe the group classification method used here and introduce the notions of conditional and partial equivalence. Then (sections 3) we significantly enhance the results of [3] and give the complete group classification of class (2). Since the case $f(x) = 1$ has a great variety of applications and has been investigated earlier by a number of authors, we collect results for this class together in section 4. Section 5 contains the proof of the main theorem on group classification of the class (2). We attempted to present our calculations in reasonable detail so that verification would be feasible. Conditional equivalence transformations are considered in section 6, where we also present four lemmas on possible local equivalence transformations between two arbitrary equations of form (2). The results of the group classification are used to find exact solutions of equations from class (2) (section 7).

2. The group classification method and additional equivalence

Let us describe the classical algorithm for group classification restricting consideration, for simplicity, to the case of one differential equation of the form

$$L^\theta(x, u_{(n)}) = L(x, u_{(n)}, \theta(x, u_{(n)})) = 0. \tag{9}$$

Here $x = (x_1, \ldots, x_l)$ denotes the independent variables, u is a dependent variable, $u_{(n)}$ is the set of all the partial derivatives of the function u with respect to x of order no greater than n, including u as the derivative of zero order. L is a fixed function of x, $u_{(n)}$ and θ. θ denotes the set of arbitrary (parametric) functions $\theta(x, u_{(n)}) = (\theta^1(x, u_{(n)}), \ldots, \theta^k(x, u_{(n)}))$ satisfying the conditions

$$S(x, u_{(n)}, \theta_{(q)}(x, u_{(n)})) = 0 \qquad S = (S_1, \ldots, S_r). \tag{10}$$

These conditions consist of r differential equations on θ, where x and $u_{(n)}$ play the roles of independent variables and $\theta_{(q)}$ stands for the set of all the partial derivatives of θ of order no greater than q. In what follows we call the functions $\theta(x, u_{(n)})$ arbitrary elements. We denote the class of equations of form (9) with the arbitrary elements θ satisfying the constraint (10) as $L|_S$.

Let the functions θ be fixed. Each one-parameter group of local point transformations that leaves equation (9) invariant corresponds to an infinitesimal symmetry operator of the form

$$Q = \xi^a(x, u)\partial_{x_a} + \eta(x, u)\partial_u$$

(here the summation over the repeated indices is understood). The complete set of such groups generates the principal group $G^{\max} = G^{\max}(L, \theta)$ of equation (9). The principal group $G^{\max}$ has a corresponding Lie algebra $A^{\max} = A^{\max}(L, \theta)$ of infinitesimal symmetry operators of equation (9). The kernel of principal groups is the group

$$G^{\ker} = G^{\ker}(L, S) = \bigcap_{\theta: S(\theta)=0} G^{\max}(L, \theta)$$

for which the Lie algebra is

$$A^{\ker} = A^{\ker}(L, S) = \bigcap_{\theta: S(\theta)=0} A^{\max}(L, \theta).$$

Let $G^{\text{equiv}} = G^{\text{equiv}}(L, S)$ denote the local transformations group preserving the form of equations from $L|_S$. (Sometimes one considers a subgroup instead the complete equivalence group.)

The problem of group classification consists in finding of all possible inequivalent cases of extensions of $A^{\max}$, i.e. in a listing all G^{equiv}-inequivalent values of θ that satisfy equation (10) and the condition $A^{\max}(L, \theta) \neq A^{\ker}$.

In the approach used here, group classification is the application of the following algorithm [2, 22]:

1. From the infinitesimal Lie invariance criterion we find the system of determining equations for the coefficients of Q. It is possible that some of the determining equations do not contain arbitrary elements and therefore can be integrated immediately. Others (i.e. the equations containing arbitrary elements explicitly) are called classifying equations. The main difficulty in group classification is the need to solve classifying equations with respect to the coefficients of the operator Q and arbitrary elements simultaneously.
2. The next step involves finding the kernel algebra $A^{\ker}$ of the principal groups of equations from $L|_S$. After decomposing the determining equations with respect to all the unconstrained derivatives of arbitrary elements, one obtains a system of partial differential equations for coefficients of the infinitesimal operator Q only. Solving this system yields the algebra $A^{\ker}$.
3. In order to construct the equivalence group G^{equiv} of the class $L|_S$ we have to investigate the local symmetry transformations of system (9), (10), considering it as a system of partial differential equations with respect to θ with the independent variables $x, u_{(n)}$. Usually one considers only transformations that can be projected on the space of the variables x and u. Although in the case where θ depends on, at most, these variables, it can be assumed that their transformations depend on θ too. After restricting ourselves

to studying the connected component of unity in G^{equiv}, we can use the Lie infinitesimal method. To find the complete equivalence group (including discrete transformations) we should use a more complicated direct method.

4. If $A^{\max}$ is an extension of $A^{\ker}$ (i.e. when $A^{\max}(L,\theta) \neq A^{\ker}$), then the classifying equations define a system of non-trivial equations for θ. Depending on their form and number, we obtain different cases of extensions of $A^{\ker}$. To completely integrate the determining equations we have to investigate a large number of such cases. In order to avoid cumbersome enumeration of possibilities in solving the determining equations we can use, for instance, algebraic methods [17–21, 27], a method which involves the investigation of compatibility of the classifying equations [23–26] or combined methods [28, 29].

The result of application of the above algorithm is a list of equations with their Lie invariance algebras. The problem of group classification is assumed to be completely solved if

(i) the list contains all the possible inequivalent cases of extensions;
(ii) all the equations from the list are mutually inequivalent with respect to the transformations from G^{equiv};
(iii) the algebras obtained are the maximal invariance algebras of the respective equations.

Such a list may include equations that are mutually equivalent with respect to local transformations which do not belong to G^{equiv}. Knowing such additional equivalences allows one to essentially simplify further investigation of $L|_S$. Constructing them can be considered as the fifth step of the algorithm of group classification. Then, the above enumeration of requirements for the resulting list of classifications can be completed by the following step:

(iv) all the possible additional equivalences between the listed equations are constructed in explicit form.

One of the ways of finding additional equivalences is based on the fact that equivalent equations have equivalent maximal invariance algebras. The second way is by systematic study of conditional and partial equivalence transformations in the class $L|_S$. Let us give a definition of such a transformation. Consider a system

$$S'(x, u_{(n)}, \theta_{(q')}(x, u_{(n)})) = 0 \qquad S' = (S'_1, \ldots, S'_{r'}) \tag{11}$$

formed by r' differential equations on θ with x and $u_{(n)}$ as independent variables. Let $G^{\text{equiv}}(L,(S,S'))$ denote the equivalence group of the subclass $L|_{S,S'}$ of $L|_S$, where the functions θ satisfy systems (10) and (11) simultaneously.

Notion 1. *We call the transformations from $G^{\text{equiv}}(L,(S,S'))$ a conditional equivalence transformation of class $L|_S$ (under the additional constraint S'). The local transformations which transform equations from $L|_{S,S'}$ to $L|_S$ are called partial equivalence transformations of the class $L|_S$ (under the additional constraint S').*

It is obvious that any conditional equivalence is a partial one under the same additional constraint and any local symmetry transformation of equation (9) for a fixed value $\theta = \theta^0(x, u_{(n)})$ is a partial equivalence transformation under the constraint $\theta = \theta^0$. The problem of description of all the possible partial equivalence transformations in the class $L|_S$ is equivalent to that of local transformations between two arbitrary equations from $L|_S$. Additional constraints on arbitrary elements never imply constraining sets of partial equivalences.

3. Results of classification

Consider a one-parameter Lie group of local transformations in (t, x, u) with an infinitesimal operator of the form $Q = \xi^t(t, x, u)\partial_t + \xi^x(t, x, u)\partial_x + \eta(t, x, u)\partial_u$, which keeps equation (2) invariant. The Lie criterion of infinitesimal invariance yields the following determining equations for ξ^t, ξ^x and η:

$$\begin{gathered}
\xi^t_x = \xi^t_u = \xi^x_u = 0 \\
D\eta_{uu} + D_u\eta_u - D_u\bigl(2\xi^x_x - \xi^t_t\bigr) + D_{uu}\eta - \frac{f_x}{f}D_u\xi^x = 0 \\
2\xi^x_x - \xi^t_t + \frac{f_x}{f}\xi^x = \frac{D_u}{D}\eta \qquad f\eta_t - K\eta_x - D\eta_{xx} = 0 \\
K\left(\frac{f_x}{f}\xi^x + \xi^x_x - \xi^t_t\right) + D\bigl(\xi^x_{xx} - 2\eta_{xu}\bigr) - 2D_u\eta_x - K_u\eta - f\xi^x_t = 0.
\end{gathered} \tag{12}$$

Investigating the compatibility of system (12) we obtain an additional equation $\eta_{uu} = 0$ without arbitrary elements. With this condition, system (12) can be rewritten in the form

$$\xi^t_x = \xi^t_u = \xi^x_u = 0 \qquad \eta_{uu} = 0 \tag{13}$$

$$2\xi^x_x - \xi^t_t + \frac{f_x}{f}\xi^x = \frac{D_u}{D}\eta \tag{14}$$

$$D\eta_{xx} + K\eta_x - f\eta_t = 0 \tag{15}$$

$$(D_uK - K_uD)\frac{\eta}{D} - K\xi^x_x - 2D_u\eta_x + D\xi^x_{xx} - f\xi^x_t - 2D\eta_{xu} = 0. \tag{16}$$

Equations (13) do not contain arbitrary elements. Integration of them yields

$$\xi^t = \xi^t(t) \qquad \xi^x = \xi^x(t, x) \qquad \eta = \eta^1(t, x)u + \eta^0(t, x). \tag{17}$$

Thus, group classification of (2) reduces to solving classifying conditions (14)–(16).

Splitting system (14)–(16) with respect to the arbitrary elements and their non-vanishing derivatives gives the equations $\xi^t_t = 0$, $\xi^x = 0$, $\eta = 0$ for the coefficients of the operators from A^{ker} of (2). As a result, the following theorem is true.

Theorem 1. *The Lie algebra of the kernel of principal groups of (2) is* $A^{\mathrm{ker}} = \langle\partial_t\rangle$.

The next step of the algorithm of group classification is finding equivalence transformations of class (2). To find these transformations, we have to investigate Lie symmetries of the system that consists of equation (2) and the additional conditions

$$f_t = f_u = 0 \qquad D_t = D_x = 0 \qquad K_t = K_x = 0.$$

Using the classical Lie approach, we find the invariance algebra of the above system that forms the Lie algebra of G^{equiv} for class (2). Thus, we obtain the following statement.

Theorem 2. *The Lie algebra of* G^{equiv} *for class (2) is*

$$A^{\mathrm{equiv}} = \langle\partial_t, \partial_x, \partial_u, t\partial_t + f\partial_f, x\partial_x - 2f\partial_f - K\partial_K, u\partial_u, f\partial_f + K\partial_K + D\partial_D\rangle. \tag{18}$$

Therefore, G^{equiv} contains the following continuous transformations:

$$\begin{array}{lll}
\tilde t = t\,\mathrm{e}^{\varepsilon_4} + \varepsilon_1 & \tilde x = x\,\mathrm{e}^{\varepsilon_5} + \varepsilon_2 & \tilde u = u\,\mathrm{e}^{\varepsilon_6} + \varepsilon_3 \\
\tilde f = f\,\mathrm{e}^{\varepsilon_4 - 2\varepsilon_5 + \varepsilon_7} & \tilde D = D\,\mathrm{e}^{\varepsilon_7} & \tilde K = K\,\mathrm{e}^{-\varepsilon_5 + \varepsilon_7}
\end{array}$$

Table 1. The case $\forall D(u)$.

N	$K(u)$	$f(x)$	Basis of $A^{\max}$
1	$\forall$	$\forall$	∂_t
2a	$\forall$	$\mathrm{e}^{\varepsilon x}$	$\partial_t,\ \varepsilon t\partial_t + \partial_x$
2b	D	$\mathrm{e}^{-2x+\gamma \mathrm{e}^{-x}}$	$\partial_t,\ \gamma t\partial_t - \mathrm{e}^x\partial_x$
2c	D	$\mathrm{e}^{-2x}(\mathrm{e}^{-x}+\gamma)^\nu$	$\partial_t,\ (\nu+2)t\partial_t - (\mathrm{e}^{-x}+\gamma)\,\mathrm{e}^x\partial_x$
2d	0	$\|x\|^\nu$	$\partial_t,\ (\nu+2)t\partial_t + x\partial_x$
2e	1	x^{-1}	$\partial_t,\ \mathrm{e}^{-t}(\partial_t - x\partial_x)$
3a	0	1	$\partial_t,\ \partial_x,\ 2t\partial_t + x\partial_x$
3b	D	e^{-2x}	$\partial_t,\ 2t\partial_t - \partial_x,\ \mathrm{e}^x\partial_x$

Here $\gamma, \nu \neq 0$, $\varepsilon = 0, 1 \bmod G^{\text{equiv}}$, $\gamma = \pm 1 \bmod G^{\text{equiv}}$.
Additional equivalence transformations:
1. 2b $\to$ 2a ($K = 0, \varepsilon = 1$): $\tilde t = t,\ \tilde x = \gamma\,\mathrm{e}^{-x},\ \tilde u = u$;
2. 2c ($\nu \neq -2$) $\to$ 2a ($K = -D/(\nu+2), \varepsilon = 1$): $\tilde t = t,\ \tilde x = (\nu+2)\ln|\mathrm{e}^{-x}+\gamma|,\ \tilde u = u$;
   2c ($\nu = -2$) $\to$ 2a ($K = -D, \varepsilon = 0$): $\tilde t = t,\ \tilde x = \ln|\mathrm{e}^{-x}+\gamma|,\ \tilde u = u$;
3. 2d ($\nu \neq -2$) $\to$ 2a ($K = -D/(\nu+2), \varepsilon = 1$): $\tilde t = t,\ \tilde x = (\nu+2)\ln|x|,\ \tilde u = u$;
   2d ($\nu = -2$) $\to$ 2a ($K = -D, \varepsilon = 0$): $\tilde t = t,\ \tilde x = \ln|x|,\ \tilde u = u$;
4. 2e $\to$ 2a ($K = -D, \varepsilon = 1$): $\tilde t = \mathrm{e}^t,\ \tilde x = \ln|x| + t,\ \tilde u = u$;
5. 3b $\to$ 3a: $\tilde t = t,\ \tilde x = \mathrm{e}^{-x},\ \tilde u = u$.

where $\varepsilon_1, \ldots, \varepsilon_7$ are arbitrary constants. For class (2) there also exists a non-trivial group of discrete equivalence transformations generated by four involutive transformations of alternating sign in the sets $\{t, D, K\}$, $\{x, K\}$, $\{u\}$ and $\{f, D, K\}$. It can be proved by the direct method that G^{equiv} coincides with the group generated by the both continuous and discrete transformations from the above list.

Theorem 3. *A complete set of inequivalent equations (2) with respect to the transformations from G^{equiv} with $A^{\max} \neq A^{\ker}$ is exhausted by cases given in tables 1–3.*

In tables 1–3 we list all possible G^{equiv}-inequivalent sets of functions $f(x)$, $D(u)$, $K(u)$ and corresponding invariance algebras. Numbers with the same Arabic figures correspond to cases that are equivalent with respect to a local equivalence transformation. Explicit formulae for these transformations are adduced after the tables. The cases numbered with different Arabic figures are inequivalent with respect to local equivalence transformations. In order to simplify the results presented, in the case $f(x) = 1$ we just use the conditional equivalence transformation $\tilde x = x - \varepsilon t$, $\tilde K = K + \varepsilon$ (the other variables are not transformed) from G_1^{equiv} (see section 4). Other conditional equivalence transformations are considered in section 6.

Below, for convenience we use double numeration *T.N* of classification cases and local equivalence transformations, where *T* denotes the number of the table and *N* the number of the case (or transformation) in table *T*. The notation 'equation *T.N*' is used for the equation of the form (2) where the parameter functions take the values from the corresponding case.

The operators from tables 1–3 form bases of the maximal invariance algebras iff the corresponding sets of the functions f, D, K are G^{equiv}-inequivalent to ones with most extensive invariance algebras. For example, in case 3.1 $(\mu, \nu) \neq (0, 0)$ and $\lambda \neq -1$ if $\nu = 0$. And in case 3.2 $(\mu, \nu) \notin \{(-2, -2), (0, 1)\}$ and $\nu \neq 0$. Similarly, in case 2.1 the constraint set on the parameters μ, ν and λ coincides with the one for case 3.1, and we can assume that $\mu = 1$ or $\nu = 1$. In case 2.2 we consider $\nu = 1$ immediately.

After analysing the results obtained, we can state the following theorem.

Theorem 4. *If an equation of form (2) is invariant with respect to a Lie algebra of dimension no less than 4 then it can be reduced by means of local transformations to one with $f(x) = 1$.*

Table 2. The case $D(u) = \mathrm{e}^{\mu u}$.

N	μ	$K(u)$	$f(x)$	Basis of $A^{\max}$
1	$\forall$	$\mathrm{e}^{\nu u}$	$\lvert x\rvert^{\lambda}$	$\partial_t,\ (\lambda\mu-\lambda\nu+\mu-2\nu)t\partial_t+(\mu-\nu)x\partial_x+\partial_u$
2	$\forall$	e^{u}	1	$\partial_t,\ \partial_x,\ (\mu-2)t\partial_t+(\mu-1)x\partial_x+\partial_u$
3	1	u	1	$\partial_t,\ \partial_x,\ t\partial_t+(x-t)\partial_x+\partial_u$
4	1	$\varepsilon\,\mathrm{e}^{u}$	$\forall$	$\partial_t,\ t\partial_t-\partial_u$
5a	1	0	$f^1(x)$	$\partial_t,\ t\partial_t-\partial_u,\ \alpha t\partial_t+(\beta x^2+\gamma_1x+\gamma_0)\partial_x+\beta x\partial_u$
5b	1	e^{u}	$f^2(x)$	$\partial_t,\ t\partial_t-\partial_u,\ \alpha t\partial_t-(\beta\,\mathrm{e}^{-x}+\gamma_1+\gamma_0\,\mathrm{e}^{x})\partial_x+\beta\,\mathrm{e}^{-x}\partial_u$
5c	1	1	x^{-1}	$\partial_t,\ x\partial_x+\partial_u,\ \mathrm{e}^{-t}(\partial_t-x\partial_x)$
6a	1	0	1	$\partial_t,\ t\partial_t-\partial_u,\ 2t\partial_t+x\partial_x,\ \partial_x$
6b	1	e^{u}	e^{-2x}	$\partial_t,\ t\partial_t-\partial_u,\ 2t\partial_t-\partial_x,\ \mathrm{e}^{x}\partial_x$
6c	1	e^{u}	$\mathrm{e}^{-2x}(\mathrm{e}^{-x}+\gamma)^{-3}$	$\partial_t,\ t\partial_t-\partial_u,\ (\mathrm{e}^{-x}+\gamma)\,\mathrm{e}^{x}\partial_x+\partial_u,\ -(\mathrm{e}^{-x}+\gamma)^2\,\mathrm{e}^{x}\partial_x+(\mathrm{e}^{-x}+\gamma)\partial_u$
6d	1	0	x^{-3}	$\partial_t,\ t\partial_t-\partial_u,\ x\partial_x-\partial_u,\ x^2\partial_x+x\partial_u$

Here $\lambda\neq0$, $\varepsilon\in\{0,1\}\bmod G^{\mathrm{equiv}}$, $\alpha, \beta, \gamma_1, \gamma_0=\mathrm{const}$ and

$$f^1(x)=\exp\left\{\int\frac{-3\beta x-2\gamma_1+\alpha}{\beta x^2+\gamma_1x+\gamma_0}\,\mathrm{d}x\right\}\qquad f^2(x)=\exp\left\{\int\frac{\beta\,\mathrm{e}^{-x}-2\gamma_0\,\mathrm{e}^{x}-\alpha}{\beta\,\mathrm{e}^{-x}+\gamma_1+\gamma_0\,\mathrm{e}^{x}}\,\mathrm{d}x\right\}.$$

Additional equivalence transformations:
1. 5b → 5a: $\tilde t=t,\ \tilde x=\mathrm{e}^{-x},\ \tilde u=u$;
2. 5c → 5a ($\alpha=\gamma_0=1,\ \beta=\gamma_1=0,\ f^1=x^{-1}$): $\tilde t=\mathrm{e}^{t},\ \tilde x=\mathrm{e}^{t}x,\ \tilde u=u$;
3. 6b → 6a: $\tilde t=t,\ \tilde x=\mathrm{e}^{-x},\ \tilde u=u$;
4. 6c → 6a: $\tilde t=t\,\mathrm{sign}(\mathrm{e}^{-x}+\gamma),\ \tilde x=1/(\mathrm{e}^{-x}+\gamma),\ \tilde u=u-\ln\lvert\mathrm{e}^{-x}+\gamma\rvert$;
5. 6d → 6a: $\tilde t=t\,\mathrm{sign}\,x,\ \tilde x=1/x,\ \tilde u=u-\ln\lvert x\rvert$.

4. Group classification for the subclass with $f(x) = 1$

Class (2) includes a subclass of equations of the general form

$$u_t=(D(u)u_x)_x+K(u)u_x \tag{19}$$

(i.e. the function f is assumed to be equal to 1 identically). Symmetry properties of equations (19) were studied in [11, 12]. But we are not aware of any paper containing a correct and exhaustive investigation of the subject. Now let us single out the results of the group classification of equations (19) from the above section.

Theorem 5. *The Lie algebra of the kernel of the principal groups of (19) is* $A_1^{\ker}=\langle\partial_t,\partial_x\rangle$.

Theorem 6. *The Lie algebra of the equivalence group* G_1^{equiv} *for the class (19) is*

$$A_1^{\mathrm{equiv}}=\langle\partial_t,\partial_x,\partial_u,u\partial_u,t\partial_x-\partial_K,2t\partial_t+x\partial_x-K\partial_K,t\partial_t-D\partial_D-K\partial_K\rangle. \tag{20}$$

Any transformation from G_1^{equiv} has the form

$$\begin{array}{lll}\tilde t=t\varepsilon_4^2\varepsilon_5+\varepsilon_1 & \tilde x=x\varepsilon_4+\varepsilon_7t+\varepsilon_2 & \tilde u=u\varepsilon_6+\varepsilon_3\\ \tilde D=D\varepsilon_5^{-1} & \tilde K=K\varepsilon_4^{-1}\varepsilon_5^{-1}-\varepsilon_7 & \end{array} \tag{21}$$

where $\varepsilon_1,\ldots,\varepsilon_7$ are arbitrary constants, $\varepsilon_4\varepsilon_5\varepsilon_6\neq0$.

Theorem 7. *The complete set of* G_1^{equiv}*-inequivalent extensions of* $A^{\max}$ *for equations (19) is given in table 4.*

Table 3. The case $D(u) = u^\mu$.

N	μ	$K(u)$	$f(x)$	Basis of $A^{\max}$
1	$\forall$	u^ν	$\|x\|^\lambda$	$\partial_t,\ (\mu+\lambda\mu-2\nu-\lambda\nu)t\partial_t+(\mu-\nu)x\partial_x+u\partial_u$
2	$\forall$	u^ν	1	$\partial_t,\ \partial_x,\ (\mu-2\nu)t\partial_t+(\mu-\nu)x\partial_x+u\partial_u$
3	$\forall$	$\ln u$	1	$\partial_t,\ \partial_x,\ \mu t\partial_t+(\mu x-t)\partial_x+u\partial_u$
4	$\forall$	εu^μ	$\forall$	$\partial_t,\ \mu t\partial_t-u\partial_u$
5a	$\forall$	0	$f^3(x)$	$\partial_t,\ \mu t\partial_t-u\partial_u,$ $\alpha t\partial_t+((1+\mu)\beta x^2+\gamma_1 x+\gamma_0)\partial_x+\beta xu\partial_u$
5b	$\forall$	u^μ	$f^4(x)$	$\partial_t,\ \mu t\partial_t-u\partial_u,$ $\alpha t\partial_t-((1+\mu)\beta\,\mathrm{e}^{-x}+\gamma_1+\gamma_0\,\mathrm{e}^x)\partial_x+\beta\,\mathrm{e}^{-x}u\partial_u$
5c	$\mu\neq-3/2$	1	x^{-1}	$\partial_t,\ \mathrm{e}^{-t}(\partial_t-x\partial_x),\ \mu x\partial_x+u\partial_u$
6a	$\mu\neq-4/3$	0	1	$\partial_t,\ \mu t\partial_t-u\partial_u,\ \partial_x,\ 2t\partial_t+x\partial_x$
6b	$\mu\neq-4/3$	u^μ	e^{-2x}	$\partial_t,\ \mu t\partial_t-u\partial_u,\ 2t\partial_t-\partial_x,\ \mathrm{e}^x\partial_x$
6c	-1	0	$\mathrm{e}^{\gamma x}$	$\partial_t,\ t\partial_t+u\partial_u,\ \partial_x-\gamma u\partial_u,\ 2t\partial_t+x\partial_x-\gamma xu\partial_u$
6d	-1	u^{-1}	$\mathrm{e}^{-2x+\gamma\mathrm{e}^{-x}}$	$\partial_t,\ t\partial_t+u\partial_u,\ \mathrm{e}^x\partial_x+\gamma u\partial_u,\ 2t\partial_t-\partial_x-\gamma\,\mathrm{e}^{-x}u\partial_u$
6e	$\mu\neq-4/3,-1$	0	$\|x\|^{-\frac{4+3\mu}{1+\mu}}$	$\partial_t,\ \mu t\partial_t-u\partial_u,\ (2+\mu)t\partial_t-(1+\mu)x\partial_x,$ $(1+\mu)x^2\partial_x+xu\partial_u$
6f	$\mu\neq-4/3,-1$	u^μ	$\frac{\mathrm{e}^{-2x}}{(\mathrm{e}^{-x}+\gamma)^{\frac{4+3\mu}{1+\mu}}}$	$\partial_t,\ \mu t\partial_t-u\partial_u,\ (2+\mu)t\partial_t+(1+\mu)(\mathrm{e}^{-x}+\gamma)\,\mathrm{e}^x\partial_x,$ $-(1+\mu)(\mathrm{e}^{-x}+\gamma)^2\,\mathrm{e}^x\partial_x+(\mathrm{e}^{-x}+\gamma)u\partial_u$
6g	$-3/2$	1	x^{-1}	$\partial_t,\ \mathrm{e}^{-t}(\partial_t-x\partial_x),\ 3x\partial_x-2u\partial_u,\ \mathrm{e}^t(x^2\partial_x-2xu\partial_u)$
7a	$-4/3$	0	1	$\partial_t,\ 4t\partial_t+3u\partial_u,\ \partial_x,\ 2t\partial_t+x\partial_x,\ x^2\partial_x-3xu\partial_u$
7b	$-4/3$	$u^{-4/3}$	e^{-2x}	$\partial_t,\ 4t\partial_t+3u\partial_u,\ 2t\partial_t-\partial_x,\ \mathrm{e}^{-x}(\partial_x+3u\partial_u),\ \mathrm{e}^x\partial_x$
8	0	u	1	$\partial_t,\ \partial_x,\ 2t\partial_t+x\partial_x-u\partial_u,\ t\partial_x-\partial_u,$ $t^2\partial_t+tx\partial_x-(tu+x)\partial_u$

Here $\mu\neq0$ for cases 4–6. $\varepsilon=0,1 \bmod G^{\text{equiv}}$, $\lambda\neq0$, $\alpha,\beta,\gamma_1,\gamma_0=\text{const}$ and

$$f^3(x)=\exp\left\{\int\frac{-(4+3\mu)\beta x-2\gamma_1+\alpha}{(1+\mu)\beta x^2+\gamma_1 x+\gamma_0}\,\mathrm{d}x\right\}$$

$$f^4(x)=\exp\left\{\int\frac{(2+\mu)\beta\,\mathrm{e}^{-x}-2\gamma_0\,\mathrm{e}^x-\alpha}{(1+\mu)\beta\,\mathrm{e}^{-x}+\gamma_1+\gamma_0\,\mathrm{e}^x}\,\mathrm{d}x\right\}.$$

Additional equivalence transformations:
1. 5b → 5a: $\tilde t=t,\ \tilde x=\mathrm{e}^{-x},\ \tilde u=u$;
2. 5c → 5a ($\alpha=\gamma_0=1,\ \beta=\gamma_1=0,\ f^1=x^{-1}$): $\tilde t=\mathrm{e}^t,\ \tilde x=\mathrm{e}^t x,\ \tilde u=u$;
3. 6b → 6a: $\tilde t=t,\ \tilde x=\mathrm{e}^{-x},\ \tilde u=u$;
4. 6c → 6a ($\mu=-1$): $\tilde t=t,\ \tilde x=x,\ \tilde u=\mathrm{e}^{\gamma x}u$;
5. 6d → 6a ($\mu=-1$): $\tilde t=t,\ \tilde x=\mathrm{e}^{-x},\ \tilde u=\mathrm{e}^{\gamma\mathrm{e}^{-x}}u$;
6. 6e → 6a: $\tilde t=t,\ \tilde x=-1/x,\ \tilde u=|x|^{-\frac{1}{1+\mu}}u$;
7. 6f → 6a: $\tilde t=t,\ \tilde x=-1/(\mathrm{e}^{-x}+\gamma),\ \tilde u=|\mathrm{e}^{-x}+\gamma|^{-\frac{1}{1+\mu}}u$;
8. 6g → 6a ($\mu=-3/2$): $\tilde t=\mathrm{e}^t,\ \tilde x=-\mathrm{e}^{-t}/x,\ \tilde u=|\mathrm{e}^t x|^{-\frac{1}{1+\mu}}u$;
9. 7b → 7a: $\tilde t=t,\ \tilde x=\mathrm{e}^{-x},\ \tilde u=u$.

5. Proof of theorem 3

Our method is based on the fact that the substitution of the coefficients of any operator from $A^{\max}\backslash A^{\ker}$ into the classifying equations results in nonidentity equations for arbitrary elements. In the problem under consideration, the procedure of looking for the possible cases mostly depends on equation (14). For any operator $Q\in A^{\max}$ equation (14) gives some equations on D of the general form

$$(au+b)D_u=cD \qquad (22)$$

Table 4. The case $f(x) = 1$.

N	$D(u)$	$K(u)$	Basis of $A^{\max}$
1	$\forall$	$\forall$	∂_t, ∂_x
2	$\forall$	0	$\partial_t, \partial_x, 2t\partial_t + x\partial_x$
3	$e^{\mu u}$	e^u	$\partial_t, \partial_x, (\mu-2)t\partial_t + (\mu-1)x\partial_x + \partial_u$
4	e^u	u	$\partial_t, \partial_x, t\partial_t + (x-t)\partial_x + \partial_u$
5	e^u	0	$\partial_t, \partial_x, t\partial_t - \partial_u, 2t\partial_t + x\partial_x$
6	u^μ	u^ν	$\partial_t, \partial_x, (\mu-2\nu)t\partial_t + (\mu-\nu)x\partial_x + u\partial_u$
7a	u^μ	0	$\partial_t, \partial_x, \mu t\partial_t - u\partial_u, 2t\partial_t + x\partial_x$
7b	u^{-2}	u^{-2}	$\partial_t, \partial_x, 2t\partial_t + u\partial_u, e^{-x}(\partial_x + u\partial_u)$
8	$u^{-4/3}$	0	$\partial_t, \partial_x, 4t\partial_t + 3u\partial_u, 2t\partial_t + x\partial_x, x^2\partial_x - 3xu\partial_u$
9	u^μ	$\ln u$	$\partial_t, \partial_x, \mu t\partial_t + (\mu x - t)\partial_x + u\partial_u$
10	1	u	$\partial_t, \partial_x, t^2\partial_t + tx\partial_x - (tu+x)\partial_u, 2t\partial_t + x\partial_x - u\partial_u, t\partial_x - \partial_u$

Here $\mu, \nu = \text{const}$. $(\mu, \nu) \neq (-2, -2), (0, 1)$ and $\nu \neq 0$ for $N = 6$. $\mu \neq -4/3$ for $N = 7$a. Case 7b can be reduced to 7a ($\mu = -2$) by means of the conditional equivalence transformation $\tilde{t} = t, \tilde{x} = e^x, \tilde{u} = e^{-x}u$.

where $a, b, c = \text{const}$. In general, for all operators from $A^{\max}$ the number k of such independent equations is no greater than 2; otherwise they form an incompatible system on D. k is an invariant value for the transformations from G^{equiv}. Therefore, there exist three inequivalent cases for the value of k: $k = 0, k = 1, k = 2$. Let us consider these possibilities in more detail, omitting cumbersome calculations.

$\boldsymbol{k = 0}$ (table 1). Here the coefficients of any operator from $A^{\max}$ must satisfy

$$\eta = 0 \qquad 2\xi^x_x - \xi^t_t + \frac{f_x}{f}\xi^x = 0 \qquad -K\xi^x_x + D\xi^x_{xx} - f\xi^x_t = 0. \tag{23}$$

Let us suppose that $K \notin \langle 1, D\rangle$. It follows from the last equation of the system (23) that $\xi^x_x = \xi^x_t = 0$. Therefore, the second equation is a nonidentity equation for f of the form $f_x = \mu f$ without fail. Solving this equation yields case 2a.

Now let $K \in \langle 1, D\rangle$, i.e. $K = \varepsilon D + \beta$ where $\varepsilon \in \{0, 1\}$, $\beta = \text{const}$. Then the last equation of (23) can be decomposed into the following ones:

$$\xi^x_{xx} = \varepsilon\xi^x_x \qquad \beta\xi^x_x + f\xi^x_t = 0.$$

The equation $(\xi^x(f_x/f + 2\varepsilon))_x = 0$ is a differential consequence of the reduced determining equations. Therefore, the condition $f_x/f + 2\varepsilon = 0$ is a classifying one.

Suppose this condition is true, i.e. $f = e^{-2\varepsilon x} \bmod G^{\text{equiv}}$. There exist three different possibilities for values of the parameters ε and β:

$$\varepsilon = 1 \quad \beta \neq 0 \qquad \varepsilon = 1 \quad \beta = 0 \qquad \varepsilon = 0 \quad \left(\text{then } \beta = 0 \bmod G^{\text{equiv}}_1\right)$$

which yield cases 2a, 3b and 3a respectively.

Let $\varepsilon = 0$ and $f_x/f \neq 0$. Then either our consideration is reduced to case 2a or $f = x^\mu \bmod G^{\text{equiv}}$ where $\mu \neq 0$. Depending on the value of the parameter β ($\beta = 0$ or $\beta \neq 0$ and then $\mu = -1$) we obtain case 2d or case 2e.

Let $\varepsilon = 1$ and $f_x/f \neq -2$. Then $\beta = 0$ and $f_x/f = (C_1 e^x + C_0)^{-1} - 2$ where we assume $C_1 \neq 0$ to exclude case 2a. Integrating the latter equation depends on whether C_0 vanishes or not, and results in cases 2b and 2c.

$\boldsymbol{k = 1}$. Here $D \in \{e^u, u^\mu, \mu \neq 0\} \bmod G^{\text{equiv}}$ and there exists $Q \in A^{\max}$ with $\eta \neq 0$.

Let us investigate the first possibility $D = \mathrm{e}^u$ (table 2). Equation (14) implies $\eta_u = 0$, i.e. $\eta = \eta(t, x)$. Therefore, equations (14)–(16) can be written as

$$2\xi^x_x - \xi^t_t + \frac{f_x}{f}\xi^x = \eta \qquad \mathrm{e}^u\eta_{xx} + K\eta_x - f\eta_t = 0$$
$$(K - K_u)\eta - K\xi^x_x - f\xi^x_t + \mathrm{e}^u\left(\xi^x_{xx} - 2\eta_x\right) = 0. \tag{24}$$

The latter equation looks like $K_u = \nu K + b\,\mathrm{e}^u + c$ with respect to K, where $\nu, b, c = \mathrm{const}$. Therefore, K must take one of the following five values.

1. $K = \mathrm{e}^{\nu u} + \varkappa_1\,\mathrm{e}^u + \varkappa_0 \bmod G^{\mathrm{equiv}}$, where $\nu \neq 0, 1$. (Here and below, $\varkappa_i = \mathrm{const}$, $i = 0, 1$.) Then $\eta = \mathrm{const}$, $\varkappa_1 = 0$ and either $\varkappa_0 = 0$ if $f \neq \mathrm{const}$ or $\varkappa_0 = 0 \bmod G^{\mathrm{equiv}}_1$ if $f = \mathrm{const}$ which implies $\xi^x_t = 0$, $\xi^t_{tt} = 0$; therefore $f = |x|^\lambda \bmod G^{\mathrm{equiv}}$ (cases 1 and 2).
2. $K = u + \varkappa_1\,\mathrm{e}^u + \varkappa_0$. In an analogous way to that in the previous case, we obtain $\varkappa_1 = 0$, $f = 1 \bmod G^{\mathrm{equiv}}$, $\varkappa_0 = 0 \bmod G^{\mathrm{equiv}}_1$ (case 3).
3. $K = u\,\mathrm{e}^u + \varkappa_1\,\mathrm{e}^u + \varkappa_0 \bmod G^{\mathrm{equiv}}$. It follows from system (24) that $\eta = 0$ for any operator from $A^{\max}$, i.e. we have a contradiction to the assumption that $\eta \neq 0$ for some operator from $A^{\max}$.
4. $K = \mathrm{e}^u + \varkappa_0$. Then $\eta^1 = \zeta^1(t)\,\mathrm{e}^{-x} + \zeta^0(t)$, $\xi^x = \sigma^1(t)\,\mathrm{e}^x + \sigma^0(t) - \zeta^1(t)\,\mathrm{e}^{-x}$. It can be proved that $\zeta^1_t = \zeta^0_t = \sigma^1_t = \xi^t_{tt} = 0$ and either $\varkappa_0 = 0$ if $f \neq \mathrm{const}$ or $\varkappa_0 = 0 \bmod G^{\mathrm{equiv}}_1$ if $f = \mathrm{const}$; therefore $\sigma^0_t = 0$. The first equation of (24) implies that the function f must satisfy l ($l = 0, 1, 2$) equations of the form
$$\frac{f_x}{f} = \frac{\beta\,\mathrm{e}^{-x} - \alpha - 2\gamma_0\,\mathrm{e}^x}{\beta\,\mathrm{e}^{-x} + \gamma_1 + \gamma_0\,\mathrm{e}^x}$$
with non-proportional sets of constant parameters $(\alpha, \beta, \gamma_0, \gamma_1)$. The values $l = 0$ and $l = 1$ correspond to cases 4 ($\varepsilon = 1$) and 5b. An additional extension of $A^{\max}$ exists for $l = 2$ in comparison with $l = 1$ iff f is a solution of the equation
$$\frac{f_x}{f} = \frac{\lambda_2\,\mathrm{e}^{-x}}{\lambda_1\,\mathrm{e}^{-x} + \lambda_0} - 2$$
where either $\lambda_2 = 0$ or $\lambda_2 = 3\lambda_1 \neq 0$. Integrating the latter equation gives cases 6b and 6c.
5. $K = \varkappa_0$. Here $\eta^1 = \zeta^1(t)x + \zeta^0(t)$, $\xi^x = \sigma^1(t)x + \sigma^0(t) + \zeta^1(t)x^2$. It follows from the compatibility of system (24) that $\eta_t = \xi^x_t = 0$ if $f \notin \{x^{-1}, 1\} \bmod G^{\mathrm{equiv}}$ or $\varkappa_0 = 0$. The values $f = x^{-1}$, $\varkappa_0 \neq 0$ result in case 5c. If $f \notin \{x^{-1}, 1\} \bmod G^{\mathrm{equiv}}$ and $\varkappa_0 = 0$, we obtain case 1 with $\nu = 0$. If $f = \mathrm{const}$ then $\varkappa_0 = 0 \bmod G^{\mathrm{equiv}}_1$. Below, $\varkappa_0 = 0$. The first equation of (24) holds when the function f is a solution of a system of l ($l = 0, 1, 2$) equations of the form
$$\frac{f_x}{f} = \frac{-3\beta x + \alpha - 2\gamma_1}{\beta x^2 + \gamma_1 x + \gamma_0}$$
with non-proportional sets of constant parameters $(\alpha, \beta, \gamma_0, \gamma_1)$. The values $l = 0$ and $l = 1$ correspond to cases 4 ($\varepsilon = 0$) and 5a. Additional extensions for $l = 2$ exist iff f is a solution of the equation
$$\frac{f_x}{f} = \frac{\lambda_2}{\lambda_1 x + \lambda_0}$$
where either $\lambda_2 = 0$ or $\lambda_2 = -3\lambda_1 \neq 0$. These possibilities result in cases 6a and 6d.

Consider the case $D = u^{\mu}$ (table 3). Equation (14) implies $\eta^0 = 0$, i.e. $\eta = \eta^1(t,x)u$. Therefore, system (14)–(16) can be written as

$$
\begin{gathered}
2\xi^x_x - \xi^t_t + \frac{f_x}{f}\xi^x = \mu\eta^1 \qquad u^{\mu}\eta^1_{xx} + K\eta^1_x - f\eta^1_t = 0 \\
(\mu K - uK_u)\eta^1 - K\xi^x_x + \left(\xi^x_{xx} - 2(\mu+1)\eta^1_x\right)u^{\mu} - f\xi^x_t = 0.
\end{gathered}
\tag{25}
$$

The latter equation looks with respect to K similarly to $uK_u = \nu K + bu^{\mu} + c$, where $\nu, b, c = \text{const}$. Therefore, K must take one of five values.

1. $K = u^{\nu} + \varkappa_1 u^{\mu} + \varkappa_0 \bmod G^{\text{equiv}}$, where $\nu \neq 0, \mu$. Equations (25) imply $\eta^1 = \text{const}$, $\xi^x = (\mu - \nu)\eta^1 x + \sigma(t)$, $\varkappa_1\xi^x_x = 0$ (therefore, $\varkappa_1 = 0$ since $\eta^1 = 0$), $f = |x|^{\lambda} \bmod G^{\text{equiv}}$, $\xi^t_t = (\mu + \lambda\mu - 2\nu - \lambda\nu)\eta^1$, $\lambda\sigma = 0$ and either $\varkappa_0 = 0$ if $\lambda \neq 0$ (case 1) or $\varkappa_0 = 0 \bmod G_1^{\text{equiv}}$ if $\lambda = 0$ (case 2).
2. $K = \ln u + \varkappa_1 u^{\mu} + \varkappa_0 \bmod G^{\text{equiv}}$. In a way analogous to that in the previous case we obtain $\varkappa_1 = 0$, $f = 1 \bmod G^{\text{equiv}}$, $\varkappa_0 = 0 \bmod G_1^{\text{equiv}}$ (case 3).
3. $K = u^{\mu}\ln u + \varkappa_1 u^{\mu} + \varkappa_0 \bmod G^{\text{equiv}}$. It follows from system (25) that $\eta = 0$ for any operator from $A^{\max}$, i.e. we have a contradiction to the assumption that $\eta \neq 0$ for some operator from $A^{\max}$.
4. $K = u^{\mu} + \varkappa_0 \bmod G^{\text{equiv}}$. Here $\eta^1 = \zeta^1(t)\,\mathrm{e}^{-x} + \zeta^0(t)$ and $\xi^x = \sigma^1(t)\,\mathrm{e}^{x} + \sigma^0(t) - (\mu + 1)\zeta^1(t)\,\mathrm{e}^{-x}$. It can be proved that $\zeta^1_t = \zeta^0_t = \sigma^1_t = \xi^t_{tt} = 0$ and either $\varkappa_0 = 0$ if $f \neq \text{const}$ or $\varkappa_0 = 0 \bmod G_1^{\text{equiv}}$ if $f = \text{const}$; therefore $\sigma^0_t = 0$. The first equation of (25) implies that the function f must satisfy l ($l = 0, 1, 2$) equations of the form
$$\frac{f_x}{f} = \frac{(\mu+2)\beta\,\mathrm{e}^{-x} - \alpha - 2\gamma_0\,\mathrm{e}^{x}}{(\mu+1)\beta\,\mathrm{e}^{-x} + \gamma_1 + \gamma_0\,\mathrm{e}^{x}}$$
with non-proportional sets of constant parameters $(\alpha, \beta, \gamma_0, \gamma_1)$. The values $l = 0$ and $l = 1$ correspond to cases 4 ($\varepsilon = 1$) and 5b. $l = 2$ iff f is a solution of the equation
$$\frac{f_x}{f} = \frac{\lambda_2\,\mathrm{e}^{-x}}{\lambda_1\,\mathrm{e}^{-x} + \lambda_0} - 2.$$
Looking for the inequivalent possibilities for integrating this equation results in cases 6b, 6d, 6f, 7b.
5. $K = \varkappa_0$. Here $\eta^1 = \zeta^1(t)x + \zeta^0(t)$, $\xi^x = \sigma^1(t)x + \sigma^0(t) + (\mu+1)\zeta^1(t)x^2$. It follows from the compatibility of system (25) that $\eta_t = \xi^x_t = 0$ if $f \not\in \{x^{-1}, 1\} \bmod G^{\text{equiv}}$ or $\varkappa_0 = 0$. The values $f = x^{-1}$, $\varkappa_0 \neq 0$ result in cases 5c and 6g. If $f \not\in \{x^{-1}, 1\} \bmod G^{\text{equiv}}$ and $\varkappa_0 = 0$, we obtain case 1 with $\nu = 0$. If $f = \text{const}$ then $\varkappa_0 = 0 \bmod G_1^{\text{equiv}}$. Below, $\varkappa_0 = 0$. The first equation of (25) holds when the function f is a solution of a system of l ($l = 0, 1, 2$) equations of the form
$$\frac{f_x}{f} = \frac{-(3\mu+4)\beta x + \alpha - 2\gamma_1}{(\mu+1)\beta x^2 + \gamma_1 x + \gamma_0}$$
with non-proportional sets of constant parameters $(\alpha, \beta, \gamma_0, \gamma_1)$. The values $l = 0$ and $l = 1$ correspond to cases 4 ($\varepsilon = 0$) and 5a. $l = 2$ iff f is a solution of the equation
$$\frac{f_x}{f} = \frac{\lambda_2}{\lambda_1 x + \lambda_0}.$$
Looking for the inequivalent possibilities for integrating this equation results in cases 6a, 6c, 6e, 7a.

Table 5. Conditional equivalence algebras.

Conditions	Basis of A^{equiv}
$K = D$	$\partial_t, \partial_x, \partial_u, u\partial_u, t\partial_t + f\partial_f, \mathrm{e}^x(\partial_x - 2f\partial_f), f\partial_f + D\partial_D$
$K = D = \mathrm{e}^u$	$\partial_t, t\partial_t + f\partial_f, \partial_x, \partial_u + f\partial_f, x\partial_x - 2f\partial_f, x^2\partial_x + x\partial_u - 3x\partial_f$
$D = \mathrm{e}^u, K = 0$	$\partial_t, t\partial_t + f\partial_f, \partial_x, \partial_u + f\partial_f, x\partial_x - 2f\partial_f, x^2\partial_x + x\partial_u - 3x\partial_f$
$D = K = u^\mu$	$\partial_t, t\partial_t + f\partial_f, \partial_x, \partial_u + \mu f\partial_f, \mathrm{e}^x(\partial_x - 2f\partial_f), \mathrm{e}^{-x}((1+\mu)\partial_x - u\partial_u + (2+\mu)f\partial_f)$
$D = u^\mu, K = 0$	$\partial_t, t\partial_t + f\partial_f, \partial_x, \partial_u + \mu f\partial_f, x\partial_x - 2f\partial_f, (1+\mu)x^2\partial_x + xu\partial_u - (4+3\mu)xf\partial_f$

$\boldsymbol{k = 2}$. The assumption of two independent equations of form (22) for D yields $D = \text{const}$, i.e. $D = 1 \bmod G^{\text{equiv}}$. $K_u \neq 0$ (otherwise, equation (2) is linear). Equations (14)–(16) can be written as

$$\begin{aligned}&2\xi^x_x - \xi^t_t + \frac{f_x}{f}\xi^x = 0 \qquad \eta_{xx} + K\eta_x - f\eta_t = 0,\\ &-K_u\eta - K\xi^x_x + \xi^x_{xx} - f\xi^x_t - 2\eta^1_x = 0.\end{aligned} \tag{26}$$

The latter equation looks similar to $(au + b)K_u = cK + d$ with respect to K, where $a, b, c, d = \text{const}$. Therefore, to within transformations from G^{equiv}, K must take one of four values:

$$K = u^\nu + \varkappa_0 \quad \nu \neq 0, 1 \qquad K = \ln u + \varkappa_0 \qquad K = \mathrm{e}^u + \varkappa_0 \qquad K = u.$$

Classification for these values is carried out in a similar way to that described above. The extensions obtained can be entered in either table 2 or table 3.

The problem of the group classification of equation (2) is exhaustively solved.

6. Additional equivalence transformations

When we impose some restrictions on arbitrary elements we can find additional equivalence transformations named conditional equivalence transformations (see notion 1). As mentioned above, the simplest way to find such equivalences between previously classified equations is based on the fact that equivalent equations have equivalent maximal invariance algebras. A more systematic way to proceed is to classify these transformations using the infinitesimal method or the direct method. Examples of conditional equivalence algebras calculated by the infinitesimal method are listed in table 5.

To find the complete collection of additional local equivalence transformations including both continuous and discrete ones, we will use the direct method. Moreover, application of this method allows us to describe all the local transformations that are possible for pairs of equations from the class under consideration. A problem of this sort was first investigated for wave equations by Kingston and Sophocleous [30–32]. Now we state a number of simple but very useful lemmas containing preliminary results in the solution of this problem. (We imply that the condition of non-singularity is satisfied for all the transformations.)

Lemma 1 [33]. *For any local transformation between two evolutionary second-order equations (i.e. equations of the form $u_t = H(t, x, u, u_x, u_{xx})$ where $H_{u_{xx}} \neq 0$) the transformation of the variable t depends only on t.*

Lemma 2. *Any local transformation between two evolutionary second-order quasi-linear equations having the form $u_t = F(t,x,u)u_{xx} + G(t,x,u,u_x)$ with $F \neq 0$ is projectable, i.e. $\tilde t = T(t)$, $\tilde x = X(t,x)$, $\tilde u = U(t,x,u)$.*

Lemma 3. *Any local transformation between two equations from class (2) is linear with respect to u: $\tilde t = T(t)$, $\tilde x = X(t,x)$, $\tilde u = U^1(t,x)u + U^0(t,x)$ and, up to transformations from $G^{\rm equiv}$, we can assume that the coefficient D is not changed.*

Lemma 4. $(U_t, U_x) \neq (0,0)$ *for a local transformation between two equations from class (2) only if* $D \in \{u^\mu, \mathrm{e}^u\} \bmod G^{\rm equiv}$.

As an example of discrete equivalence transformations we can give the involution

$$\tilde t = t \qquad \tilde x = -x \qquad \tilde u = u + \alpha x$$

in the couple of equations

$$f(x)u_t = (\mathrm{e}^u u_x)_x + \alpha\,\mathrm{e}^u u_x \qquad \text{and} \qquad \mathrm{e}^{-\alpha x} f(-x)u_t = (\mathrm{e}^u u_x)_x + \alpha\,\mathrm{e}^u u_x.$$

Moreover, this transformation is a discrete invariance transformation for the equation

$$g(x)\,\mathrm{e}^{-\alpha x/2}u_t = (\mathrm{e}^u u_x)_x + \alpha\,\mathrm{e}^u u_x$$

iff g is an even function.

We also investigated some transformations into other classes of reaction–diffusion equations. Using the discrete transformation $\tilde t = t$, $\tilde x = -x$, $\tilde u = u + x/2$ we can reduce the equation

$$\mathrm{e}^{-x/2}u_t = \mathrm{e}^u\left(u_{xx} + u_x^2 + u_x\right)$$

to the reaction–diffusion equation from the classification of Dorodnitsyn [10]:

$$\tilde u_t = (\mathrm{e}^{\tilde u}\tilde u_{\tilde x})_{\tilde x} - \tfrac14\,\mathrm{e}^{\tilde u}.$$

7. Exact solutions

We now turn to the presentation of some exact solutions for (2). Using our classification with respect to all the possible local transformations (i.e. not only with respect to ones from $G^{\rm equiv}$), first we can obtain solutions of simpler equations (e.g. 6a from tables 2 or 3) by means of the classical Lie–Ovsiannikov algorithm or non-classical methods. Then we transform them to solutions of more complicated equations (such as 6b, 6c).

Let us note that the equations with $f = 1$ are well investigated and that most of the exact solutions given below have been constructed before (see citations in [36]). However, to the best of our knowledge, there exist no works containing a systematic study of all the possible Lie reductions in this class, as well as exhaustive consideration of the integrability and exact solutions of the corresponding reduced equations. That is why we have decided to implement the relevant Lie reduction algorithm independently, especially since it is not a difficult problem.

So, let us consider equation 2.6a:

$$u_t = (\mathrm{e}^u u_x)_x. \tag{27}$$

Let us recall that for (27) the basis of $A^{\max}$ is formed by the operators

$$Q_1 = \partial_t \qquad Q_2 = t\partial_t - \partial_u \qquad Q_3 = \partial_x \qquad Q_4 = x\partial_x + 2\partial_u.$$

The only non-zero commutators of these operators are $[Q_1, Q_2] = Q_1$ and $[Q_3, Q_4] = Q_3$. Therefore $A^{\max}$ is a realization of the algebra $2A_{2.1}$ [34]. All the possible inequivalent (with

Table 6. Reduced ODEs for (27). $\alpha \neq 0$, $\varepsilon = \pm 1$, $\delta = \operatorname{sign} t$.

N	Subalgebra	Ansatz $u =$	ω	Reduced ODE
1	$\langle Q_3\rangle$	$\varphi(\omega)$	t	$\varphi' = 0$
2	$\langle Q_4\rangle$	$\varphi(\omega) + 2\ln\lvert x\rvert$	t	$\varphi' = 2e^{\varphi}$
3	$\langle Q_1\rangle$	$\varphi(\omega)$	x	$(e^{\varphi})'' = 0$
4	$\langle Q_2\rangle$	$\varphi(\omega) - \ln\lvert t\rvert$	x	$(e^{\varphi})'' = -\delta$
5	$\langle Q_1 + \varepsilon Q_3\rangle$	$\varphi(\omega)$	$x - \varepsilon t$	$(e^{\varphi})'' = -\varepsilon\varphi'$
6	$\langle Q_2 + \varepsilon Q_3\rangle$	$\varphi(\omega) - \ln\lvert t\rvert$	$x - \varepsilon\ln\lvert t\rvert$	$(e^{\varphi})'' = -\delta(\varepsilon\varphi' + 1)$
7	$\langle Q_1 + \varepsilon Q_4\rangle$	$\varphi(\omega) + 2\varepsilon t$	$xe^{-\varepsilon t}$	$(e^{\varphi})'' = -\varepsilon\omega\varphi' + 2\varepsilon$
8	$\langle Q_2 + \alpha Q_4\rangle$	$\varphi(\omega) + (2\alpha - 1)\ln\lvert t\rvert$	$x\lvert t\rvert^{-\alpha}$	$(e^{\varphi})'' = \delta(-\alpha\omega\varphi' + 2\alpha - 1)$

respect to inner automorphisms) one-dimensional subalgebras of $2A_{2.1}$ [35] are exhausted by the ones listed in table 6 along with the corresponding ansätze and the reduced ODEs.

We succeeded in solving the equations 6.1–6.5. Thus we have the following solutions of (27):

$$u = \ln|c_1x + c_0| \qquad u = \ln\left(\frac{-x^2}{2t} + \frac{c_1x + c_0}{t}\right)$$

$$u = \varphi(x - \varepsilon t) \qquad \text{where} \quad \int \frac{\mathrm{e}^{\varphi}}{c_1 - \varepsilon\varphi}\,\mathrm{d}\varphi = \omega + c_0.$$

Using these we can construct solutions for cases 2.6b–2.6d easily. For example, the transformation 2.4 yields the corresponding solutions for the more complicated and interesting equation

$$\frac{\mathrm{e}^x}{(\gamma\,\mathrm{e}^x + 1)^3}u_t = (\mathrm{e}^u u_x)_x + \mathrm{e}^u u_x$$

having the localized density (case 2.6c)

$$u = \ln|c_1 + c_0(\mathrm{e}^{-x} + \gamma)| \qquad u = \ln\left(-\frac{1}{2t(\mathrm{e}^{-x} + \gamma)} - \frac{c_1}{t} + c_0\frac{\mathrm{e}^{-x} + \gamma}{t}\right).$$

$\mu = -1$ is a singular value of the parameter μ for case 3.6a. So, the equation

$$u_t = \left(\frac{u_x}{u}\right)_x \tag{28}$$

is distinguished by the reduction procedure. It is remarkable that cases 3.6c and 3.6d are reduced exactly to equation (28). The invariance algebra of (28) is generated by the operators

$$Q_1 = \partial_t \qquad Q_2 = t\partial_t + u\partial_u \qquad Q_3 = \partial_x \qquad Q_4 = x\partial_x - 2u\partial_u$$

and is a realization of the algebra $2A_{2.1}$ too. The reduced ODEs for equation (28) are listed in table 7. After integrating cases 7.1–7.4 we obtain the following solutions of (28):

$$u = c_0\,\mathrm{e}^{c_1x} \qquad u = \frac{2c_1^2t}{\cos^2 c_1(x + c_0)} \qquad u = \frac{2tc_0c_1^2\,\mathrm{e}^{c_1x}}{(1 - c_0\,\mathrm{e}^{c_1x})^2}$$

$$u = \frac{c_1}{-\varepsilon + c_0\,\mathrm{e}^{c_1(x-\varepsilon t)}} \qquad u = \frac{\varepsilon}{x - \varepsilon t + c_0} \qquad u = \frac{2t}{(x + c_1)^2 + c_0t^2}.$$

Analogously to the previous case, we obtain by means of transformations 3.5 exact solutions

Table 7. Reduced ODEs for (28). $\alpha \neq 0$, $\varepsilon = \pm 1$.

N	Subalgebra	Ansatz $u =$	ω	Reduced ODE
1	$\langle Q_3\rangle$	$\varphi(\omega)$	t	$\varphi' = 0$
2	$\langle Q_4\rangle$	$\varphi(\omega)x^{-2}$	t	$\varphi' = 2$
3	$\langle Q_1\rangle$	$\varphi(\omega)$	x	$(\varphi^{-1}\varphi')' = 0$
4	$\langle Q_2\rangle$	$\varphi(\omega)t$	x	$(\varphi^{-1}\varphi')' = \varphi$
5	$\langle Q_1 + \varepsilon Q_3\rangle$	$\varphi(\omega)$	$x - \varepsilon t$	$(\varphi^{-1}\varphi')' = -\varepsilon\varphi'$
6	$\langle Q_2 + \varepsilon Q_3\rangle$	$\varphi(\omega)t$	$x - \varepsilon \ln\|t\|$	$(\varphi^{-1}\varphi')' = -\varepsilon\varphi' + \varphi$
7	$\langle Q_1 + \varepsilon Q_4\rangle$	$\varphi(\omega)\,\mathrm{e}^{-2\varepsilon t}$	$x\,\mathrm{e}^{-\varepsilon t}$	$(\varphi^{-1}\varphi')' = -\varepsilon\omega\varphi' - 2\varepsilon\varphi$
8	$\langle Q_2 + \alpha Q_4\rangle$	$\varphi(\omega)t\|t\|^{-2\alpha}$	$x\|t\|^{-\alpha}$	$(\varphi^{-1}\varphi')' = -\alpha\omega\varphi' + (1-2\alpha)\varphi$

Table 8. Reduced ODEs for (29). $\mu \neq 0, -1$, $\alpha \neq 0$, $\varepsilon = \pm 1$, $\delta = \operatorname{sign} t$.

N	Subalgebra	Ansatz $u =$	ω	Reduced ODE
1	$\langle Q_3\rangle$	$\varphi(\omega)$	t	$\varphi' = 0$
2	$\langle Q_4\rangle$	$\varphi(\omega)\|x\|^{2/\mu}$	t	$\varphi' = 2\mu^{-2}(2+\mu)\varphi^{\mu+1}$
3	$\langle Q_1\rangle$	$\varphi(\omega)$	x	$(\varphi^{\mu}\varphi')' = 0$
4	$\langle Q_2\rangle$	$\varphi(\omega)\|t\|^{-1/\mu}$	x	$(\varphi^{\mu}\varphi')' = -\delta\mu^{-1}\varphi$
5	$\langle Q_1 + \varepsilon Q_3\rangle$	$\varphi(\omega)$	$x - \varepsilon t$	$(\varphi^{\mu}\varphi')' = -\varepsilon\varphi'$
6	$\langle Q_2 + \varepsilon Q_3\rangle$	$\varphi(\omega)\|t\|^{-1/\mu}$	$x - \varepsilon \ln\|t\|$	$(\varphi^{\mu}\varphi')' = -\delta\varepsilon\varphi' - \delta\mu^{-1}\varphi$
7	$\langle Q_1 + \varepsilon Q_4\rangle$	$\varphi(\omega)\,\mathrm{e}^{2\varepsilon\mu^{-1}t}$	$xe^{-\varepsilon t}$	$(\varphi^{\mu}\varphi')' = -\varepsilon\omega\varphi' + 2\mu^{-1}\varepsilon\varphi$
8	$\langle Q_2 + \alpha Q_4\rangle$	$\varphi(\omega)\|t\|^{(2\alpha-1)/\mu}$	$x\|t\|^{-\alpha}$	$(\varphi^{\mu}\varphi')' = \delta\mu^{-1}(2\alpha-1)\varphi - \delta\alpha\omega\varphi'$

of equation 3.6d in the following forms:

$$u = c_0\,\mathrm{e}^{(c_1-\gamma)\mathrm{e}^{-x}} \qquad u = \frac{2c_1^2 t\,\mathrm{e}^{-\gamma\mathrm{e}^{-x}}}{\cos^2 c_1(\mathrm{e}^{-x}+c_0)} \qquad u = \frac{2tc_0c_1^2\,\mathrm{e}^{(c_1-\gamma)\mathrm{e}^{-x}}}{(1-c_0\,\mathrm{e}^{c_1\mathrm{e}^{-x}})^2}$$

$$u = \frac{c_1\,\mathrm{e}^{-\gamma\mathrm{e}^{-x}}}{-\varepsilon + c_0\,\mathrm{e}^{c_1(\mathrm{e}^{-x}-\varepsilon t)}} \qquad u = \frac{\varepsilon\,\mathrm{e}^{-\gamma\mathrm{e}^{-x}}}{\mathrm{e}^{-x} - \varepsilon t + c_0} \qquad u = \frac{2t\,\mathrm{e}^{-\gamma\mathrm{e}^{-x}}}{(\mathrm{e}^{-x}+c_1)^2 + c_0t^2}.$$

Another example of an equation with a localized density is given by case 3.6f. To look for exact solutions of it, first we reduce it to equation 3.6a:

$$u_t = (u^{\mu}u_x)_x\,. \tag{29}$$

As in the previous cases, the invariance algebra of (29)

$$A^{\max} = \langle Q_1 = \partial_t,\, Q_2 = t\partial_t - \mu^{-1}u\partial_u,\, Q_3 = \partial_x,\, Q_4 = x\partial_x + 2\mu^{-1}u\partial_u\rangle$$

is a realization of the algebra $2A_{2.1}$. The result of reduction (29) under inequivalent subalgebras of $A^{\max}$ is written down in table 8.

For some of the reduced equations we can construct the general solutions. For others we succeeded in finding only particular solutions. These solutions are the following:

$$u = |c_1x + c_0|^{\frac{1}{\mu+1}} \qquad u = (c_0 - \varepsilon\mu(x-\varepsilon t))^{\frac{1}{\mu}} \qquad u = \left(-\frac{\mu}{\mu+2}\frac{(x+c_0)^2}{2t} + c_1|t|^{-\frac{\mu}{\mu+2}}\right)^{\frac{1}{\mu}}$$

$$u = \left(-\frac{\mu}{\mu+2}\frac{(x+c_0)^2}{2t} + c_1(x+c_0)^{\frac{\mu}{\mu+1}}|t|^{-\frac{\mu(2\mu+3)}{2(\mu+1)^2}}\right)^{\frac{1}{\mu}}.$$

All the results of tables 7, 8 as well as the solutions constructed can be extended to equations 3.6b–3.6g using the local equivalence transformations. So for the equation

$$\frac{\mathrm{e}^{-2x}}{(\mathrm{e}^{-x}+\gamma)^{\frac{4+3\mu}{1+\mu}}}u_t = (u^\mu u_x)_x + u^\mu u_x \tag{30}$$

(case 3.6f) the transformations 3.7 yield exact solutions in the form

$$u = |c_0(\mathrm{e}^{-x}+\gamma) - c_1|^{\frac{1}{\mu+1}}$$

$$u = \left(c_0 + \frac{\varepsilon\mu}{\mathrm{e}^{-x}+\gamma} + \varepsilon^2\mu t\right)^{\frac{1}{\mu}} |\mathrm{e}^{-x}+\gamma|^{-\frac{1}{\mu+1}}$$

$$u = \left(-\frac{\mu}{\mu+2}\frac{1}{2t}\left(c_0 - \frac{1}{\mathrm{e}^{-x}+\gamma}\right)^2 + c_1|t|^{-\frac{\mu}{\mu+2}}\right)^{\frac{1}{\mu}} |\mathrm{e}^{-x}+\gamma|^{-\frac{1}{\mu+1}}$$

$$u = \left(-\frac{\mu}{\mu+2}\frac{1}{2t}\left(c_0 - \frac{1}{\mathrm{e}^{-x}+\gamma}\right)^2 + c_1\left(c_0 - \frac{1}{\mathrm{e}^{-x}+\gamma}\right)^{\frac{\mu}{\mu+1}} |t|^{-\frac{\mu(2\mu+3)}{2(\mu+1)^2}}\right)^{\frac{1}{\mu}} |\mathrm{e}^{-x}+\gamma|^{-\frac{1}{\mu+1}}.$$

A number of exact solutions were constructed for equations from class (19) ($f = 1$) by means of non-classical methods. Starting from them and using local transformations of conditional equivalence, we can obtain non-Lie exact solutions for more complicated equations (cases 6b, 6c, ...).

Amerov [38] and King [39] suggested looking for solutions of the equation $u_t = (u^{-1/2}u_x)_x$ (equation 3.6a, $\mu = -1/2$) in the form $u = (\varphi^1(x)t + \varphi^0(x))^2$ where the functions $\varphi^1(x)$ and $\varphi^0(x)$ satisfy the system of ODEs $\varphi^1_{xx} = (\varphi^1)^2$, $\varphi^0_{xx} = \varphi^0\varphi^1$. A particular solution of this system is

$$\varphi^1 = \frac{6}{x^2} \qquad \varphi^0 = \frac{c_1}{x^2} + \frac{c_2}{x^3}.$$

And the corresponding solution of equation 3.6f with $\mu = -1/2$ can be written down as

$$u = (6t + c_1' + c_2\,\mathrm{e}^{-x})^2(\mathrm{e}^{-x}+\gamma)^6.$$

8. Conclusion

In this paper, the group classification in the class of equations (2) is performed completely. The main results on classification are collected in tables 1–3 where we list inequivalent cases of extensions with the corresponding Lie invariance algebras. Among the equations presented there exist ones which have the density f localized in the space of x and are invariant with respect to more abundant Lie algebras than A^{ker}. Following the tables, we write down all the additional equivalence transformations, reducing some equations from our classification to others of simpler forms. (In fact, the equations of (2) have been classified with respect to two different equivalence relations generated by either the equivalence group or the set of all possible transformations.) For a number of equations from the list of the reduced ones we construct optimal systems of inequivalent subalgebras, corresponding Lie ansätze and exact invariant solutions. By means of additional equivalence transformations the solutions obtained are transformed to the ones for the more interesting and complicated equations with localized densities.

We describe equivalence transformations within a class of PDEs (2) using the infinitesimal and direct methods. The direct method enables us to find the group of all possible local equivalence transformations (i.e. not only continuous ones) in the whole class (2) as well as all the conditional equivalence transformations. Moreover, we begin to solve the general equivalence problem for any pair of equations from class (2) with respect to the local transformations (lemmas 1–4).

We plan to continue investigations of this subject. For the class under consideration we plan to perform classification of potential and non-classical (conditional) symmetries and finish studying all the possible partial equivalence transformations. We also intend to investigate the existence, localization and asymptotic properties of solutions of initial and boundary-value problems for nonlinear convection–diffusion equations.

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